Kernel Function Occurring in Subsonic Unsteady Potential Flow

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Abstract

THIS paper deals with the Kernel function of the integral equation relating the pressure to the normal-wash distribution in unsteady potential subsonic flow. Exact solutions of the involved integrals of the Kernel function are given in terms of new functions. Efficient and accurate numerical evaluation of these functions are described.

Nomenclature

$Ci(x)$	= cosine integral function
$f(x)$	= first auxiliary function of the trigonometric integral functions, $Ci(x)\sin(x) - si(x)\cos(x)$
$g(x)$	= second auxiliary function of the trigonometric integral functions, $-Ci(x)\cos(x) - si(x)\sin(x)$
$I_\mu(x)$	= modified Bessel function of first kind and order μ
$J_\mu(x)$	= Bessel function of first kind and order μ
$K(\ )$	= Kernel function relating normal wash at point x,y,z to unit pressure difference at point ξ, η, ζ
$K_\mu(x)$	= modified Bessel function of second kind and order μ
k	= reduced frequency, $\omega r/U$
$L_\mu(x)$	= modified Struve function of order μ
M	= mach number
q	= freestream dynamic pressure, $\rho U^2/2$
R	= $(x_0^2 + \beta^2 r^2)^{1/2}$
r	= $[(y-\eta)^2 + (z-\zeta)^2]^{1/2}$
$Si(x)$	= sine integral function
$si(x)$	= $Si(x) - \pi/2$
U	= freestream velocity in x direction
u	= $(MR - x_0)/\beta^2 r$
$w(x,y,z)$	= normal velocity at point x,y,z
x,y,z	= coordinates of the normal-wash point
x_0,y_0,z_0	= $x-\xi$, $y-\eta$, $z-\zeta$
β	= $(1-M^2)^{1/2}$
$\Gamma(x)$	= gamma function
γ	= local dihedral angle
Δp	= pressure difference
ξ,η,ζ	= coordinates of the doublet point
ρ	= freestream air density

Subscripts

r	= receiving point
s	= sending point
μ	= $0,1,2,3,\ldots$
ν	= $1/2, 3/2, 5/2, \ldots$

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Statement of the Problem

The relationship between the pressure and the normal-wash distribution in unsteady potential subsonic flows reads[1]

$$w(x,y,z)/U = (1/8\pi) \times \iint \left[\Delta p(\xi,\eta,\zeta)K(x,y,z,\xi,\eta,\zeta;k,M)/(qr^2)\right] \mathrm{d}\xi\, \mathrm{d}\eta \qquad (1)$$

The Kernel function K of the integral equation [Eq. (1)] can be written as[2]

$$K = \exp(-i\omega x_0/U)(K_1T_1 + K_2T_2) \qquad (2)$$

T_1 and T_2 are geometric relations and are given by

$$T_1 = \cos(\gamma_r - \gamma_s) \qquad (3a)$$

$$T_2 = (z_0\cos\gamma_r - y_0\sin\gamma_r)(z_0\cos\gamma_s - y_0\sin\gamma_s)/r^2 \qquad (3b)$$

The functions K_1 and K_2 of Eq. (2) were evaluated by Landahl[3] and are given by

$$K_1 = Mre^{-iku}/\left[R^2(1+u^2)^{1/2}\right] + N_{3/2} \qquad (4a)$$

$$K_2 = -ikM^2r^2e^{-iku}/\left[R^2(1+u^2)^{1/2}\right] - Mre^{-iku} \times \left[(1+u^2)\beta^2r^2 + 2R^2 + MRru\right]\Big/\left[R^3(1+u^2)^{3/2}\right] - 3N_{5/2} \qquad (4b)$$

where

$$N_\nu = \int_u^\infty \left[e^{-ikv}(1+v^2)^\nu\right] \mathrm{d}v; \qquad \nu = 3/2 \text{ and } 5/2 \qquad (5)$$

The first attempt to evaluate the integrals [Eq. (5)] was made by Watkins et al.,[2] who approximated the algebraic part of the integrand by a four term exponential approximation. Later on Laschka[4] presented an analytical solution in terms of infinite series. This series solution has the defect of very slow convergence, limiting its practical application to small values of the arguments. For this reason, Laschka[4] also presented a numerical solution similar to that of Watkins et al.,[2] including an 11-term exponential approximation. The integrals performed using the Laschka approximation have an accuracy of the order of three digits. A similar numerical solution was presented by Dat and Malfois.[5] The accuracy of this solution and the computation effort involved are similar to the Laschka approximation. The exponential approximation of the algebraic part was used again by Desmarais,[6] who used the least-square technique to minimize the error in the approximation. The precision of the integrals evaluated using the Desmarais approximation with 12 terms is of the order of three to four digits. Asymptotic expansion of the Laschka series was given by Ueda[7] for large values of the arguments. In the following, an analytical solution is presented for the evaluation of the integrals [Eq. (5)]. The problem is first transformed to the solution of a differential equation. The function solutions of the differential equation are then given. Being analytical, the solution's precision can be attained to any desired accuracy; furthermore, the computation effort involved is much less than the numerical method previously used.

Consider, now, the integral

$$N_\nu(u) = \int_u^\infty \left[e^{-ikv}/(1+v^2)^\nu\right] \mathrm{d}v$$

$$= N_{R_\nu}(u) + iN_{I_\nu}(u), \qquad \nu = 1/2, 3/2, 5/2, \ldots \tag{6}$$

where k is assumed to be a non-negative constant parameter, and we notice that it is sufficient to consider only non-negative values of the argument u due to the symmetric properties of the integrands. Defining the function $F_\nu(u)$ as

$$F_\nu(u) = N_{R_\nu}(u)\sin(ku) + N_{I_\nu}(u)\cos(ku) \tag{7}$$

and making use of Eqs. (6), we obtain a differential equation,

$$F_\nu''(u) + k^2F(u) = -k/(1+u^2)^\nu \tag{8}$$

with the following boundary conditions,

$$F_\nu(0) = -\left[(\pi)^{1/2}/2\right]\Gamma(1-\nu)(k/2)^\mu\left[I_{-\mu}(k) - L_{-\mu}(k)\right] \tag{9a}$$

$$F_\nu'(0) = \left[(\pi)^{1/2}/\Gamma(\nu)\right](k/2)^\mu kK_\mu(k) \tag{9b}$$

so that the problem is transformed to the solution of Eq. (8) subject to the boundary conditions [Eqs. (9)]. The integrals [Eq. (5)] can then be evaluated from

$$N_{R_\nu}(u) = F_\nu(u)\sin(ku) + (1/k)F_\nu'(u)\cos(ku) \tag{10a}$$

$$N_{I_\nu}(u) = F_\nu(u)\cos(ku) - (1/k)F_\nu'(u)\sin(ku) \tag{10b}$$

Recurrence Relations of the Function $F_\nu(u)$

Recurrence relations of the function $F_\nu(u)$ and its derivative can be obtained by integrating by parts of Eq. (6) twice and making use of Eq. (7) to read,

$$4\nu(1+\nu)F_{\nu+2}(u) = k^2F_\nu(u) + 2\nu(2\nu+1)F_{\nu+1}(u) + k/(1+u^2)^\nu \tag{11}$$

$$4\nu(1+\nu)F_{\nu+2}'(u) = k^2F_\nu'(u) + 2\nu(2\nu+1)F_{\nu+1}'(u) - 2\nu uk/(1+u^2)^{\nu+1} \tag{12}$$

Evaluation of the Function $F_\nu(u)$

Applying the Laplace transform on Eq. (8) and using the recurrence relation [Eq. (12)] and the recurrence relations of the Bessel functions we obtain an integral representation of the function $F_\nu(u)$ as

$$F_\nu(u) = -k(\pi)^{1/2}/\left[2^\mu\Gamma(\nu)\right]\int_0^\infty t^\mu J_\mu(t)e^{-ut}/(t^2+k^2)\,\mathrm{d}t$$

$$\mu = (2\nu-1)/2 = 0,1,2,3,\ldots \tag{13}$$

Solution for $u > 1$

Expanding the Bessel function in Eq. (13) in its ascending power series, we obtain,

$$F_{1/2}(u) = -f(ku) - \sum_{n=1}^{\infty} c_n(ku)(k/2)^{2n}/(n!)^2$$

$$= -f(ku)I_0(k) + \sum_{n=0}^{\infty}(-)^n a_n(k)(2n!)/(2u)^{2n+1} \tag{14}$$

$$F_{3/2}(u) = k\sum_{n=1}^{\infty} c_n(ku)(k/2)^{2n-1}/\left[(n-1)!\,n!\right]$$

$$= kf(ku)I_1(k) - k\sum_{n=0}^{\infty}(-)^n b_n(k)(2n)!/(2u)^{2n+1} \tag{15}$$

where

$$c_n(ku) = f(ku) + \sum_{m=1}^{n}(-)^m(2m-2)!/(ku)^{2m-1} \tag{16a}$$

$$a_n(k) = \sum_{m=0}^{\infty}(k/2)^{2m+1}/\left[(m+n+1)!\right]^2 \tag{16b}$$

$$b_n(k) = \sum_{m=0}^{\infty}(k/2)^{2m}/\left[(m+n)!(m+n+1)!\right] \tag{16c}$$

Solution for $u < 1$

Expanding the exponential term in Eq. (13) in its power series solution, making use of the recurrence relations of Bessel and Sturve functions, and using Hankel transform operations, we obtain,

$$F_{1/2}(u) = -(\pi/2)\left[I_0(k) - L_0(k)\right]\cos(ku) + K_0(k)\sin(ku) + \sum_{n=1}^{\infty}(-)^n(ku)^{2n}c_n/(2n)! \tag{17}$$

where

$$c_n = \sum_{m=1}^{n}(2m-3)^2(2m-5)^2\ldots 3^21^2/k^{2m-1}$$

and

$$F_{3/2}(u) = (\pi k/2)\left[I_{-1}(k) - L_{-1}(k)\right]\cos(ku) + kK_1(k)\sin(ku) + \sum_{n=1}^{\infty}(-)^n(ku)^{2n}c_n/(2n)! \tag{18}$$

where

$$c_n = \sum_{m=1}^{n}(2m-1)(2m-3)^2(2m-5)^2\ldots 3^21^2/k^{2m-1}$$

The full paper contains a more detailed derivation of the equations, a solution convergent for all values of k and u, and also considers direct numerical integration of Eq. (13). In addition, a complete set of formulas that preserve a precision of six digits for the evaluation of the functions $F_{1/2}(u)$ and $F_{3/2}(u)$ and their derivatives are given. The full paper also provides rational Chebychev approximations for the computation of the auxiliary trigonometric integral functions, the modified Bessel and Sturve functions, and asymptotic expansions for large values of k of the function $F_\nu(u)$.

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